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Quenched Large Deviation Principle for the Overlap of a *p*-Spins System

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In this note, we prove a quenched large deviation result for the overlap of a p-spins interaction system at high temperature. The rate function of the large deviation principle is proved to be deterministic, and some of its basic properties are studied. Our result is based on a pure state result for a multidimensional p-spins system combined with a careful application of the Gärtner–Ellis Theorem

KEY WORDS: Multidimensional p-spins interaction system; large deviation principle.

1. INTRODUCTION

This paper is concerned with the usual p-spins system, that is a system of N elementary spins represented by a configuration $\sigma = (\sigma_1,..., \sigma_N) \in \Sigma_N = \{-, 1; 1\}^N$, whose coordinates are governed by a random mean field type interaction of range p. This kind of physical model is generally represented by a Gibbs measure $G = G_N$ of the form $dG_N = [Z_N^*]^{-1} \exp(-\beta H_N^*)\, d\mu_N$, where μ_N is the uniform measure on Σ_N, H^* is a Hamiltonian defined by

$$-H_N^*(\sigma) = u_N \sum_{(i_1,..., i_p) \in A_N} g_{i_1,..., i_p} \sigma_{i_1} \cdots \sigma_{i_p},$$

with

$$u_N = \left(\frac{p!}{2N^{p-1}} \right)^{\frac{1}{2}}$$

$$A_N = \{(i_1,..., i_p);\ 1 \leqslant i_1 < \cdots < i_p \leqslant N\},$$

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and where $g=\{g_{i_1,\dots,i_p};\,(i_1,\dots,i_p)\in A_N\}$ is a family of independent standard Gaussian random variables. The random variable Z_N^* is then a normalizing constant converting G_N into a (random) probability measure on Σ_N, and β stands for the inverse of the temperature parameter. We will denote by $\langle f\rangle_*$ the average of a function f defined on Σ_N with respect to G_N, that is

$$\langle f\rangle_* = [Z_N^*]^{-1}\sum_{\sigma\in\Sigma_N} f(\sigma)\exp(-\beta H_N^*(\sigma)). \tag{1}$$

After Parisi's analysis of the SK model (see, e.g., ref. 1), it became clear that one of the central objects of study in the spin glasses theory was the overlap $R^{1,2}$ defined by

$$R^{1,2}=\frac{1}{N}\sum_{i\leqslant N}\sigma_i^1\sigma_i^2,$$

where σ^1,σ^2 are taken as two independent configurations under G_N. Physicists are generally mostly concerned by the behavior of this quantity in the low temperature regime, and by the related phenomenon of non-self-averaging (see, e.g., refs. 2 and 3 for rigorous results, as well as ref. 1 for a general discussion on the topic). On the other hand, at high temperature, the self averaging for $R^{1,2}$ takes place, and one can expect to understand in great details the limiting system defined by G_N when $N\to\infty$, especially for quantities like the overlap. While this kind of sharp results is mostly of mathematical interest, it is still a challenging problem to try to understand the detailed behavior of G_N. Among the papers adressing that issue for the SK model, let us mention for instance ref. 4 for the fluctuations of the free energy, ref. 5 for some identities on the overlap distribution, and ref. 6 for a detailed description of many fine asymptotic properties of G_N at high temperature.

It is then a natural, though relatively unadressed question (see however Bovier *et al.*,[7] where a central limit theorem for the fluctuations of the normalization constant Z_N^* of the p-spins model is established, following the stochastic calculus approach initiated in ref. 8) to try to get this kind of sharp results for the p-spins model. The aim of this note is then to make one step in that direction, and we will obtain a quenched (with repect to the disorder g) large deviation principle for $R^{1,2}$ as $N\to\infty$, namely we will prove that if β is small enough, then almost surely, for any Borelian subset C of $\mathbb{R}$, we will have

$$\begin{aligned}-\inf\{\Lambda^*(x);\,x\in C^\circ\} &\leqslant \liminf_{N\to\infty}\frac{1}{N}\log(\langle \mathbf{1}_C(R^{1,2})\rangle_*)\\ &\leqslant \limsup_{N\to\infty}\frac{1}{N}\log(\langle \mathbf{1}_C(R^{1,2})\rangle_*)\leqslant -\inf\{\Lambda^*(x);\,x\in\bar C\},\end{aligned}$$

with a complete (though implicit) characterization of the rate function Λ^*, which in particular happens to be a deterministic function. This constitutes a generalization of ref. 9, giving the same type of result for the SK model. However, a first difference with respect to ref. 9 can already be stressed at this stage: while ref. 9 is mainly concerned with the study of the multidimensional SK model, we will try to delve deeper into the precise statement and proof of the large deviation principle itself, which will lead us to combine both spin glasses and large deviation techniques. Let us also mention some of the possible generalizations of this work: first we believe that the case of a p-spin model with external field, whose study is an ongoing work started in ref. 10, can be handled with the same methods as ours, just changing the reference measure μ we will consider at Section 2. A more challenging result would be to obtain the annealed large deviation principle for $R^{1,2}$, but the powerful methodology elaborated in ref. 11 seems to be hard to adapt to our high dimensional case.

As in the SK case, the large deviation result for $R^{1,2}$ can be reduced to a L^2-limit computation for the overlap of a 2-dimensional p-spin system. Since there is no additional difficulty to treat the same kind of problem for a d-dimensional system, we will perform these computations at Section 2. Then, a detailed statement and proof of the main result will be given at Section 3. In the sequel, all the constants, that can change from line to line, will be denoted by κ, and we will stress their dependence on the parameters of the system by writing κ_d, $\kappa_{d,p}$, etc.

2. MULTIDIMENSIONAL *p*-SPINS MODEL

As in ref. 9, our large deviation principle will heavily rely on a pure state type result for a multidimensional p-spins model. We will describe precisely this model in the next subsection, and then derive our result.

2.1. Description of the Model

Let S be the ball of radius $d^{1/2}$ in $\mathbb{R}^d$. The state space of the multidimensional p-spins model is

$$S^N = \{\sigma = (\sigma_1, \ldots, \sigma_N);\, \sigma_j \in S,\, j \leqslant N\}.$$

For $j \leqslant N$, we will write $\sigma_j = \{\sigma_j(u);\, u \leqslant d\}$. The Hamiltonian under consideration on S^N will be of the form

$$-H_N(\sigma) = u_N \sum_{(i_1,\ldots,i_p) \in A_N} \sum_{u \leqslant d} g_{i_1,\ldots,i_p} \sigma_{i_1}(u) \cdots \sigma_{i_p}(u), \tag{2}$$

where u_N and A_N are defined as before. Let μ be a probability measure on S, and set

$$Z_N = \int_{S^N} \exp(-\beta H_N(\sigma))\, d\mu^{\otimes N}(\sigma).$$

Let $n \geqslant 1$. We will consider in this section the Gibbs measure defined on test functions $f: (S^N)^n \to \mathbb{R}$ by

$$\langle f \rangle = Z_N^{-n} \int_{S^N} f(\sigma^1, \dots, \sigma^n) \exp\left(-\sum_{l \leqslant n} \beta H_N(\sigma^l) \right) \prod_{l \leqslant n} d\mu^{\otimes N}(\sigma^l). \tag{3}$$

Notice that the dependence on n is omitted in the left hand side of the above expression.

For $N \geqslant 2$, let us split now this integral into an integral for the $N-1$ first spins on one hand, and the last spins on the other hand, a basic step known as the cavity method. For $n \geqslant 1$ and a given test function $f: (S^N)^n \to \mathbb{R}$, by some easy algebraic manipulations, we obtain the following relation:

$$\langle f \rangle = Z_N^{-n} \langle \mathbf{Av}\, f \mathcal{E}_n \rangle_-.$$

In this last expression, we have used the following conventions: for a function $\varphi: S^n \to \mathbb{R}$ we set

$$\mathbf{Av}\, \varphi = \int_{S^n} \varphi(\varepsilon_1, \dots, \varepsilon_n) \prod_{l \leqslant n} d\mu(\varepsilon^l).$$

For a given $\sigma \in S^N$, we write $\sigma = (\rho, \varepsilon)$, where $\rho \in S_{N-1}$ and $\varepsilon \in S$. For $u \leqslant d$, we denote then by $T(\rho(u))$ the vector

$$T(\rho(u)) = \{\eta_I(u);\, I = (i_1, \dots, i_{p-1}) \in B_{N-1}\},$$

where

$$B_N = \{(i_1, \dots, i_{p-1});\, 1 \leqslant i_1 < \cdots < i_{p-1} \leqslant N\}$$
$$\eta_I(u) = \sigma_{i_1}(u) \cdots \sigma_{i_{p-1}}(u), \qquad I \in B_{N-1}.$$

Set also

$$g(T(\rho(u))) = \beta u_N \sum_{I \in B_{N-1}} g_I \eta_I(u).$$

Eventually, denote by β_- the inverse temperature given by

$$\beta_- = \left(\frac{N-1}{N}\right)^{\frac{p-1}{2}} \beta, \tag{4}$$

and $\langle . \rangle_-$ the averaging with respect to G_N at temperature β_-. Then $\mathscr{E}_n$ and Z_N are given by

$$\mathscr{E}_n = \exp\left(\sum_{l \leqslant n} \sum_{u \leqslant d} g(T(\rho^l(u)))\, \varepsilon^l(u) \right)$$

$$Z_N = \langle \mathbf{Av}\, \mathscr{E}_1 \rangle_- = \left\langle \int_S \exp\left(\sum_{u \leqslant d} g(T(\rho(u)))\, \varepsilon(u) \right) d\mu(\varepsilon) \right\rangle_-$$

Let us end this introduction by an elementary estimate that will be useful later on:

Lemma 2.1. For $u, v \leqslant d$ and $l, l' \leqslant n$, set

$$R^{l,l'}(u, v) = \frac{1}{N} \sum_{i \leqslant N} \sigma_i^l(u)\, \sigma_i^{l'}(v).$$

Then there exists a positive constant $\kappa_{d,p}$ such that

$$\left| u_N^2 \sum_{I \in B_{N-1}} \eta_I^l(u)\, \eta_I^{l'}(v) - \frac{p}{2} (R^{l,l'}(u, v))^{p-1} \right| \leqslant \frac{\kappa_{d,p}}{N}.$$

Proof. We have

$$(R^{l,l'}(u, v))^{p-1} = \sum_{i_1, \ldots, i_{p-1} \leqslant N} \sigma_{i_1}^l(u)\, \sigma_{i_1}^{l'}(v) \cdots \sigma_{i_{p-1}}^l(u)\, \sigma_{i_{p-1}}^{l'}(v).$$

Hence

$$\left| \frac{1}{N^{p-1}} \sum_{I \in B_{N-1}} \eta_I^l(u)\, \eta_I^{l'}(v) - \frac{1}{(p-1)!} (R^{l,l'}(u, v))^{p-1} \right|$$
$$\leqslant \frac{1}{(p-1)!\, N^{p-1}} \sum_{I \in C_N} |\eta_I^l(u)\, \eta_I^{l'}(v)|,$$

where

$$C_N = \{1 \leqslant i_1 \cdots \leqslant i_{p-1} \leqslant N;\ \text{There exists } j \neq k \text{ satisfying } i_j = i_k\}.$$

But $\frac{|C_N|}{N^{p-1}} \leqslant \frac{c_p}{N}$ for a constant $c_p > 0$, and $\eta_I^l(u) \leqslant d^{(p-1)/2}$. Thus

$$\left| \frac{1}{N^{p-1}} \sum_{I \in B_{N-1}} \eta_I^l(u)\, \eta_I^{l'}(v) - \frac{1}{(p-1)!} (R^{l,l'}(u,v))^{p-1} \right| \leqslant \frac{c_d d^{p-1}}{N},$$

from which our lemma is then easily deduced. ∎

Let us also recall that, using some trivial combinatorics, we get

$$|(p-1)!|\, B_{N-1}\, | - N^{p-1}| \leqslant \kappa_p N^{p-2}, \tag{5}$$

for a constant $\kappa_p > 0$.

2.2. Decoupling the Last Spin

In this subsection, we will transform our original measure into a measure under which the last spin is made independent of the others through a (carefully chosen) continuous path. The consequences of this construction will be a crucial step to get the limiting behavior of the overlap. The continuous path we will consider is given, for $u \leqslant d$, $t \in [0, 1]$, by the following transformation of $g(T(\rho(u)))$:

$$\begin{aligned} g_t(T(\rho(u))) &= t^{1/2} g(T(\rho(u))) + (1-t)^{1/2}\, \beta u_N \sum_{I \in B_{N-1}} Y_I(u) \\ &\quad + \frac{\beta^2 u_N^2\, |B_{N-1}|}{2} (1-t) \sum_{v \leqslant d} \varepsilon(v)(m(u,v) - k(u,v)), \end{aligned} \tag{6}$$

where $\{m(u,v), k(u,v); u, v \leqslant d\}$ are two given symmetric matrices whose precise entries will be determined later on. In the last expression, $\{Y_I(u);$ $I \in B_{N-1}, u \leqslant d\}$ is a family of centered Gaussian random variables, independent of the disorder g, with covariance structure given by

$$E[Y_I(u)\, Y_J(v)] = k(u,v)\, \mathbf{1}_{I=J}, \qquad I, J \in B_{N-1}, \quad u, v \leqslant d. \tag{7}$$

In the right hand side of (6) as well as in the sequel of the paper, $|D|$ will denote the size of a given finite set D. Notice that the second term in the right hand side of (6) is the one that allows us to make the Nth spin independent of the others for $t = 0$, the last term of this same expression being a correction term taking into account the fact that we will consider both independent and correlated copies of the Gibbs measure (3).

For $t \in [0, 1]$ and $n \geqslant 1$, define

$$\langle f \rangle_t = Z_t^{-n} \langle \mathbf{Av}\, f \mathscr{E}_{n,t} \rangle_-,$$

where

$$\mathscr{E}_{n,t} = \exp\left(\sum_{l \leqslant n} \sum_{u \leqslant d} g_t(T(\rho^l(u)))\, \varepsilon^l(u) \right)$$

$$Z_t = \langle \mathbf{Av}\, \mathscr{E}_{1,t} \rangle_-.$$

Set also $\nu_t(f) = E[\langle f \rangle_t]$ for $t \in [0, 1]$, and $\nu(f) = \nu_1(f)$. The first useful identity related to (6) is obtained in the following proposition:

Proposition 2.2. Set $k^{l,l'}(u, v) = k(u, v)$ if $l \neq l'$, and $k^{l,l}(u, v) = m(u, v)$. Then, for all bounded measurable functions $f\colon (S^N)^n \to \mathbb{R}$, we have

$$\begin{aligned} \nu_t'(f) = \beta^2 u_N^2 \sum_{I \in B_{N-1}} \sum_{u, v \leqslant d} \Bigg(& \sum_{1 \leqslant l < l' \leqslant n} \nu_t(f \varepsilon^l(u)\, \varepsilon^{l'}(v)[\eta_I^l(u)\, \eta_I^{l'}(v) - k^{l,l'}(u, v)]) \\ & - 2n \sum_{l \leqslant n} \nu_t(f \varepsilon^l(u)\, \varepsilon^{n+1}(v)[\eta_I^l(u)\, \eta_I^{n+1}(v) - k^{l,n+1}(u, v)]) \\ & - n\nu_t(f \varepsilon^{n+1}(u)\, \varepsilon^{n+1}(v)[\eta_I^{n+1}(u)\, \eta_I^{n+1}(v) - k^{n+1,n+1}(u, v)]) \\ & + n(n+1)\, \nu_t(f \varepsilon^{n+1}(u)\, \varepsilon^{n+2}(v)[\eta_I^{n+1}(u)\, \eta_I^{n+2}(v) - k^{n+1,n+2}(u, v)]) \Bigg). \end{aligned} \tag{8}$$

Proof. Notice that

$$\begin{aligned} \frac{\partial_t \mathscr{E}_{n,t}}{\mathscr{E}_{n,t}} = \sum_{l \leqslant n} \sum_{u \leqslant d} \varepsilon^l(u) \Bigg(& \frac{1}{2t^{1/2}} g(T(\rho^l(u))) - \frac{1}{2(1-t)^{1/2}} \beta u_N \sum_{I \in B_{N-1}} Y_I(u) \\ & - \frac{\beta^2 u_N^2}{2} |B_{N-1}| \sum_{v \leqslant d} \varepsilon^l(v)(m(u, v) - k(u, v)) \Bigg). \end{aligned}$$

Thus $\nu_t'(f) = A - B - C$, with

$$\begin{aligned} A = \frac{1}{2t^{1/2}} \sum_{u \leqslant d} \mathbf{E} \Bigg[& Z_t^{-n} \sum_{l \leqslant n} \langle \mathbf{Av}\, f \varepsilon^l(u)\, g(T(\rho^l(u)))\, \mathscr{E}_{n,t} \rangle_- \\ & - n Z_t^{-(n+1)} \langle \mathbf{Av}\, f \varepsilon^{n+1}(u)\, g(T(\rho^{n+1}(u)))\, \mathscr{E}_{n+1,t} \rangle_- \Bigg], \end{aligned}$$

B defined by

$$B = \frac{\beta u_N}{2(1-t)^{1/2}} \sum_{u \leqslant d} \sum_{I \in B_{N-1}} \mathbf{E}[Z_t^{-n} \sum_{l \leqslant n} \langle \mathbf{Av}\, f \varepsilon^l(u)\, Y_I(u)\, \mathscr{E}_{n,t} \rangle_- \\ - n Z_t^{-(n+1)} \langle \mathbf{Av}\, f \varepsilon^{n+1}(u)\, Y_I(u)\, \mathscr{E}_{n+1,t} \rangle_-],$$

and

$$C = \frac{\beta^2 u_N^2\, |B_{N-1}|}{2} \sum_{u,\, v \leqslant d} \mathbf{E}\Bigg[Z_t^{-n} \sum_{l \leqslant n} \langle \mathbf{Av}\, f \varepsilon^l(u)\, \varepsilon^l(v)\, \Delta(u,v)\, \mathscr{E}_{n,t} \rangle_- \\ - n Z_t^{-(n+1)} \langle \mathbf{Av}\, f \varepsilon^{n+1}(u)\, \varepsilon^{n+1}(v)\, \Delta(u,v)\, \mathscr{E}_{n+1,t} \rangle_- \Bigg],$$

where we have set $\Delta(u, v) = m(u, v) - k(u, v)$.

Let us simplify the term B: for $r \geqslant 1$, a smooth function $F\colon \mathbb{R}^r \to \mathbb{R}$ growing at most exponentially, and a Gaussian family $(g, g_1, \ldots, g_r)$, we have

$$\mathbf{E}[gF(g_1, \ldots, g_r)] = \sum_{k \leqslant r} \mathbf{E}[g_k g]\, \mathbf{E}[\partial_{x_k} F(g_1, \ldots, g_r)]. \tag{9}$$

Furthermore, for $I \in B_{N-1}$, $u \leqslant d$,

$$\partial_{Y_I(u)} \mathscr{E}_{n,t} = \sum_{l \leqslant n} (1-t)^{1/2}\, \beta u_N \varepsilon^l(u)\, \mathscr{E}_{n,t}.$$

Hence, integrating first with respect to the variables $Y_I(u)$ and using relation (9), we get (recall that the covariance structure of Y_I is given by (7))

$$B = \frac{\beta^2 u_N^2}{2} \sum_{I \in B_{N-1}} \sum_{u,\, v \leqslant d} k(u,v)\, \mathbf{E}\Bigg[Z_t^{-n} \sum_{l,\, l' \leqslant n} \langle \mathbf{Av}\, f \varepsilon^l(u)\, \varepsilon^{l'}(v)\, \mathscr{E}_{n,t} \rangle_- \\ - n Z_t^{-(n+1)} \sum_{l \leqslant n} \langle \mathbf{Av}\, f \varepsilon^l(u)\, \varepsilon^{n+1}(v)\, \mathscr{E}_{n+1,t} \rangle_- \\ - n Z_t^{-(n+1)} \sum_{l \leqslant n+1} \langle \mathbf{Av}\, f \varepsilon^l(u)\, \varepsilon^{n+1}(v)\, \mathscr{E}_{n+1,t} \rangle_- \\ + n(n+1)\, Z_t^{-(n+2)} \langle \mathbf{Av}\, f \varepsilon^{n+1}(u)\, \varepsilon^{n+2}(v)\, \mathscr{E}_{n+2,t} \rangle_- \Bigg].$$

The same kind of calculations can be performed for A, and then putting together A, B and C, we get the announced result. ∎

Let us assume now that for all $u, v \leqslant d$, we have $|k(u, v)| \leqslant d^{p-1}$ and $|m(u, v)| \leqslant d^{p-1}$. Then the following consequence of the last proposition holds true.

Proposition 2.3. Let $f: (S^N)^n \to \mathbb{R}_+$ be a non-negative function. Then, for N large enough,

$$v_t(f) \leqslant \exp(5\beta^2 d^{p+2} n^2 p)\, v(f).$$

Proof. Since $|\eta_I(u)| \leqslant d^{(p-1)/2}$ for all $l \leqslant n$, $I \in B_{N-1}$, $u \leqslant d$, a simple estimate performed on the expression (8) gives, for N large enough,

$$\begin{aligned} v_t'(f) &\leqslant -(3n^2+2n)\, \beta^2 u_N^2 d^{p+2}\, |B_{N-1}|\, v_t(f) \\ &\leqslant -5\beta^2 d^{p+2} n^2 p v_t(f), \end{aligned}$$

where we have used the fact that $\lim_{N\to\infty} u_N^2\, |B_{N-1}| = \frac{p}{2}$, and hence $u_N^2\, |B_{N-1}| \leqslant p$ for N large enough. Integrating the last relation between t and 1, we get

$$\log(v(f)) - \log(v_t(f)) \geqslant -5\beta^2 d^{p+2} n^2 p(1-t) \geqslant -5\beta^2 d^{p+2} n^2 p,$$

which yields the desired result. ∎

2.3. Study of a Quadratic Form

Besides the overlap $R^{1,2}$, it will be useful in the sequel to consider the quantity

$$R(u, v) = \frac{1}{N} \sum_{i \leqslant N} \sigma_i(u)\, \sigma_i(v), \qquad u, v \leqslant d.$$

Let $Q = \{q(u, v);\, u, v \leqslant d\}$ be a symmetric matrix such that $Q^{p-1} \equiv \{[q(u, v)]^{p-1};\, u, v \leqslant d\}$ defines a positive quadratic form. Let S be a d-dimensional matrix satisfying the same property. When $N \to \infty$, $(R^{1,2}, R)$ will tend to a couple (Q, S) as above, solution to the equation

$$q(u, v) = \mathbf{E}[Z^{-2} \mathbf{Av}\, x^1(u)\, x^2(v)\, \mathscr{F}_2] \tag{10}$$

$$s(u, v) = \mathbf{E}[Z^{-1} \mathbf{Av}\, x(u)\, x(v)\, \mathscr{F}_1], \tag{11}$$

where $\mathscr{F}_n$ and Z are defined by

$$\mathscr{F}_n = \exp\left(\sum_{l \leqslant n}\left(\left(\frac{p}{2}\right)^{1/2} \beta \sum_{w \leqslant d} Y(w)\, x^l(w)\right.\right.$$
$$\left.\left. + \frac{\beta^2 p}{4} \sum_{w, z \leqslant d} x^l(w)\, x^l(z)([s(w, z)]^{p-1} - [q(w, z)]^{p-1})\right)\right) \tag{12}$$

and

$$Z = \mathbf{Av}\, \mathscr{F}_1.$$

In all the preceding formulae, $\{Y(u); u \leqslant d\}$ stands for a Gaussian vector with correlation matrix Q^{p-1}. We will write the system of equations (10) and (11) under the form

$$(Q, S) = \Phi(Q, S) = (\Phi^{(1)}(Q, S), \Phi^{(2)}(Q, S)). \tag{13}$$

Note that we impose the positivity of the quadratic form Q^{p-1} instead of Q, as could be expected. This can be justified by the following fact: if $\{\alpha(u); u \leqslant d\}$ is a given vector in $\mathbb{R}^d$, we will have

$$L^2 - \lim_{N \to \infty} \left\langle \sum_{u, v \leqslant d} \alpha(u)\, \eta_I^1(u)\, \eta_I^2(v)\, \alpha(v) \right\rangle = \sum_{u, v \leqslant d} \alpha(u)[q(u, v)]^{p-1} \alpha(v).$$

But

$$\left\langle \sum_{u, v \leqslant d} \alpha(u)\, \eta_I^1(u)\, \eta_I^2(v)\, \alpha(v) \right\rangle = \left\langle \sum_{u \leqslant d} \alpha(u)\, \eta_I(u) \right\rangle^2 \geqslant 0.$$

The aim of this subsection is to prove the following claim:

Proposition 2.4. If β is small enough, there is a unique pair (Q, S) such that $|q(u, v)| \vee |s(u, v)| \leqslant d$ for all $u, v \leqslant d$, solution to the system (10) and (11).

Proof. We will divide this proof in 2 steps

Step 1. The equation (10) defining Q is of the form $q(u, v) = \Phi_{u,v}^{(1)}(Q, S)$ for a given $\Phi_{u,v}^{(1)}: \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \to \mathbb{R}$, and $\Phi_{u,v}^{(1)}$ is again of the form

$$\Phi_{u,v}^{(1)}(Q, S) = \Psi_{u,v}(Q^{p-1}, S^{p-1})$$

for a function $\Psi_{u,v}: \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \to \mathbb{R}$.

Next, it is easily seen that $|\Phi^{(1)}_{u,v}(Q, S)| \leqslant d$ for all $u, v \leqslant d$ and Q, S defined as above. Furthermore, to show our claim, it is enough to show that

$$|\partial_{q(w,z)} \Phi^{(1)}_{u,v}(Q, S)| \vee |\partial_{s(w,z)} \Phi^{(1)}_{u,v}(Q, S)| \leqslant \kappa_{d,p} \beta^2,$$

for all $u, v, w, z \leqslant d$. Hence, by boundedness of $\Phi_{u,v}$, it is enough to show that $\|\nabla \Psi_{u,v}(K, M)\| \leqslant \kappa_{d,p} \beta^2$ for all K, M such that

$$\sup\{|k(u, v)| \vee |m(u, v)|;\, u, v \leqslant d\} \leqslant d.$$

We will first investigate the behavior of the derivative of $\Psi_{u,v}$ with respect to the coefficients $k(u, v)$.

Step 2. Let $K = \{k(u, v);\, u, v \leqslant d\}$ and $\hat{K} = \{\hat{k}(u, v);\, u, v \leqslant d\}$ be two positive symmetric matrices, and $Y, \hat{Y}$ two independent Gaussian vectors with respective covariance matrices K and $\hat{K}$. Set, for $w, z \leqslant d$, $t \in [0, 1]$,

$$Y_t(w) = t^{1/2} Y(w) + (1-t)^{1/2}\, \hat{Y}(w)$$
$$k_t(w, z) = t k(w, z) + (1-t)\, \hat{k}(w, z),$$

and related to those quantities, set

$$\mathcal{F}_{n,t} = \exp\left(\sum_{l \leqslant n} \left(\left(\frac{p}{2}\right)^{1/2} \beta \sum_{w \leqslant d} Y_t(w)\, x^l(w) \right.\right.$$
$$\left.\left. + \frac{\beta^2 p}{4} \sum_{w,z \leqslant d} x^l(w)\, x^l(z)([m(w, z)]^{p-1} - [k_t(w, z)]^{p-1})\right)\right),$$

and

$$q(u, v) = \mathbf{E}[Z_t^{-2} \mathbf{Av}\, x^1(u)\, x^2(v)\, \mathcal{F}_{2,t}],$$

where

$$Z_t = \mathbf{Av}\, \mathcal{F}_{1,t}.$$

The same kind of computations as in the proof of relation (8) show that $\psi'_{u,v}(t) = A - B + C$, with

$$A = \beta \left(\frac{p}{2t}\right)^{\frac{1}{2}} \sum_{w \leqslant d} \mathbf{E}\left[Z_t^{-2} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^2(w)\, Y(w)\, \mathcal{F}_{2,t} \right.$$
$$\left. - Z_t^{-3} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^3(w)\, Y(w)\, \mathcal{F}_{3,t} \right],$$

the quantity B defined by

$$B = \beta \left(\frac{p}{2(1-t)} \right)^{\frac{1}{2}} \sum_{w \leqslant d} \mathbf{E} \Big[Z_t^{-2} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^2(w)\, \hat{Y}(w)\, \mathscr{F}_{2,t} - Z_t^{-3} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^3(w)\, \hat{Y}(w)\, \mathscr{F}_{3,t} \Big],$$

and

$$C = \frac{\beta^2 p}{2} \sum_{w,z \leqslant d} \mathbf{E} \Big[Z_t^{-2} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^2(w)\, x^2(z)\, \ell(u,v)\, \mathscr{F}_{2,t} - Z_t^{-3} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^3(w)\, x^3(z)\, \ell(u,v)\, \mathscr{F}_{3,t} \Big],$$

with $\ell(u,v) = [k(u,v)]^{p-1} - [\hat{k}(u,v)]^{p-1}$. Integrating by parts with respect to $Y(w)$ in A gives

$$A = \frac{\beta^2 p}{2} \sum_{w,z \leqslant d} \mathbf{E}[Z_t^{-2} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^2(w)\, x^2(z)\, \mathscr{F}_{2,t} - Z_t^{-2} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^2(w)\, x^1(z)\, \mathscr{F}_{2,t} - 3Z_t^{-3} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^3(w)\, x^3(z)\, \mathscr{F}_{3,t} - 3Z_t^{-4} \mathbf{Av}\, x^1(u)\, x^2(v)\, x^3(w)\, x^4(z)\, \mathscr{F}_{4,t}],$$

and hence

$$|A| \leqslant 4\beta^2 p d^4.$$

The same kind of bound can be obtained for B. A direct estimate for C is given by

$$|C| \leqslant 2pd^{p+1}\beta^2.$$

Thus, for all $t \in [0,1]$,

$$|\Psi'_{u,v}(t)| \leqslant 10pd^{p+1}\beta^2. \tag{14}$$

This bound is of course independent of t, and by continuity, taking the value of $\Psi'_{u,v}$ at $t=1$ gives us the desired bound on the derivative of $\Psi_{u,v}(K,M)$ with respect to K. The same kind of calculations can be leaded for the derivatives with respect to M, and for the equation (11) defining S, which ends the proof. ∎

2.4. Limit Behavior of the Overlap

In this subsection, for $u, v \leqslant d$ and $l, l' \leqslant n$, the following quantities will be under consideration:

$$R(u, v) = \frac{1}{N} \sum_{i \leqslant N} \sigma_i(u)\, \sigma_i(v), \qquad R^{l,l'}(u, v) = \frac{1}{N} \sum_{i \leqslant N} \sigma_i^l(u)\, \sigma_i^{l'}(v). \tag{15}$$

In the remainder of the section, we will assume that for all $u, v \leqslant d$, $k(u, v)$ (resp. $m(u, v)$) can be written as $[q(u, v)]^{p-1}$ (resp. $[s(u, v)]^{p-1}$) for some symmetric matrices Q, S such that $|q(u, v)| \vee |s(u, v)| \leqslant d^{1/2}$. Using the results of Section 2.2, one gets the

Proposition 2.5. There exists a strictly positive constant $\kappa_{d,p}$ such that for all test functions $f\colon (S^N)^n \to \mathbb{R}$, we have

$$|v_t'(f)| \leqslant \kappa_{d,p} \beta^2 \left\{ v_t^{1/2} \left(\sum_{u, v \leqslant d} (R^{1,2}(u, v) - q(u, v))^2 \right) \right.$$
$$\left. + v_t^{1/2} \left(\sum_{u, v \leqslant d} (R(u, v) - s(u, v))^2 \right) + \frac{1}{N} \right\} v_t^{1/2}(f).$$

Proof. Going back to the expression of $v_t'(f)$ given by (8), let us study the term

$$M^{l,l'} \equiv u_N^2 \sum_{u, v \leqslant d} \sum_{I \in B_{N-1}} \varepsilon^l(u)\, \varepsilon^{l'}(v) (\eta_I^l(u)\, \eta_I^{l'}(v) - [q^{l,l'}(u, v)]^{p-1}).$$

Setting

$$a_I^{l,l'}(u, v) = u_N^2 (\eta_I^l(u)\, \eta_I^{l'}(v) - [q^{l,l'}(u, v)]^{p-1}),$$

by Cauchy–Schwarz inequality, we have

$$|M^{l,l'}| = \left| \sum_{u, v \leqslant d} \varepsilon^l(u)\, \varepsilon^{l'}(v) \sum_{I \in B_{N-1}} a_I^{l,l'}(u, v) \right|$$
$$\leqslant \left(\sum_{u, v \leqslant d} (\varepsilon^l(u)\, \varepsilon^{l'}(v))^2 \right)^{1/2} \left(\sum_{u, v \leqslant d} \left(\sum_{I \in B_{N-1}} a_I^{l,l'}(u, v) \right)^2 \right)^{1/2}$$
$$\leqslant d^2 \left(\sum_{u, v \leqslant d} \left(\sum_{I \in B_{N-1}} a_I^{l,l'}(u, v) \right)^2 \right)^{1/2}.$$

Furthermore,

$$\sum_{I \in B_{N-1}} a_I^{l,l'}(u,v) = u_N^2 \sum_{I \in B_{N-1}} \eta_I^l(u)\, \eta_I^{l'}(v) - u_N^2\, |B_{N-1}|\, [q^{l,l'}(u,v)]^{p-1}.$$

Now, by inequality (5), we have

$$\left| u_N^2\, |B_{N-1}|\, [q^{l,l'}(u,v)]^{p-1} - \frac{p}{2} [q^{l,l'}(u,v)]^{p-1} \right| \leqslant \frac{\kappa_p d^{p-1}}{N},$$

and combining this fact with the result of Lemma 2.1, it is easily seen that

$$|M^{l,l'}| \leqslant \kappa_{d,p} \left(\sum_{u,v \leqslant d} \left([R^{l,l'}(u,v)]^{p-1} - [q^{l,l'}(u,v)]^{p-1} + \frac{1}{N} \right)^2 \right)^{1/2}.$$

Moreover, if $|q^{l,l'}(u,v)| \leqslant d$, we get

$$|[R^{l,l'}(u,v)]^{p-1} - [q^{l,l'}(u,v)]^{p-1}| \leqslant p d^{p-2}\, |R^{l,l'}(u,v) - q^{l,l'}(u,v)|,$$

and thus

$$|M^{l,l'}| \leqslant \kappa_{d,p} \left\{ \left(\sum_{u,v \leqslant d} (R^{l,l'}(u,v) - q^{l,l'}(u,v))^2 \right)^{1/2} + \frac{1}{N} \right\}.$$

The other terms in (8) can be treated using the same method, and applying Cauchy–Schwarz's inequality to $v_t'(f)$ yields the announced result. ∎

Lemma 2.6. Let Q be the solution to (10) and f^- be a test function from S_{N-1}^n to $\mathbb{R}$. Then, for all $u, v \leqslant d$,

$$|\nu_0(f^- \varepsilon^1(u)\, \varepsilon^2(v)) - \nu_0(f^-)\, q(u,v)| \leqslant \frac{\kappa_{d,p}}{N} \|f^-\|_\infty.$$

Proof. It is easily seen that

$$\nu_0(f^- \varepsilon^1(u)\, \varepsilon^2(v)) = \mathbf{E}[\langle f^- \rangle_-]\, \mathbf{E}[W_N^{-2} \mathbf{Av}\, x^1(u)\, x^2(v)\, \mathscr{G}_2],$$

with $Y_N(u) = u_N \sum_{I \in B_{N-1}} Y_I(u)$ and

$$\mathscr{G}_n = \exp\Bigg(\sum_{l \leqslant n} \Bigg(\sum_{w \leqslant d} Y_N(w)\, x^l(w) + \frac{\beta^2 u_N^2}{2} |B_{N-1}| \sum_{w,z \leqslant d} x^l(w)\, x^l(z) ([s(w,z)]^{p-1} - [q(w,z)]^{p-1}) \Bigg) \Bigg), \tag{16}$$

the random variable W_N being defined by

$$W_N = \mathbf{Av}\, \mathcal{G}_1.$$

Note that Y_N is a Gaussian random vector with covariance matrix given by

$$\mathbf{E}[Y_N(u)\, Y_N(v)] = s_N^2 [q(u, v)]^{p-1},$$

where, using relation (5),

$$s_N^2 = u_N^2\, |B_{N-1}| = \frac{p}{2} + \delta(N), \tag{17}$$

and $\delta(N) \leqslant \frac{\kappa_p}{N}$. Consider now the function $\psi_{u,v}$ defined on $\mathbb{R}_+$ by

$$\psi_{u,v}(s) = \mathbf{E}[W_s^{-2} \mathbf{Av}\, x^1(u)\, x^2(v)\, \mathcal{G}_2(s)],$$

where

$$\mathcal{G}_n(s) = \exp\Bigg(s \sum_{l \leqslant n} \Bigg(\sum_{w \leqslant d} Y(w)\, x^l(w) + \frac{\beta^2 s^2}{2} |B_{N-1}| \sum_{w, z \leqslant d} x^l(w)\, x^l(z)([s(w, z)]^{p-1} - [q(w, z)]^{p-1}) \Bigg)\Bigg)$$

and

$$W_s = \mathbf{Av}\, \mathcal{E}_1(s),$$

Y being a Gaussian random vector with covariance Q^{p-1}. Then

$$\nu_0(f^- \varepsilon^1(u)\, \varepsilon^2(v)) = \mathbf{E}[\langle f^- \rangle_-]\, \psi_{u,v}(s_N).$$

It can be shown, as for relation 8 and Proposition 2.4, that $\psi_{u,v}$ is a C^1 function with bounded derivative on $\mathbb{R}_+$. By relation (17), we will also get $|\psi_{u,v}(s_N) - \psi_{u,v}((p/2)^{1/2})| \leqslant \frac{\kappa_p}{N}$. But Q is the solution to the equation

$$q(u, v) = \psi_{u,v}\left(\left(\frac{p}{2}\right)^{\frac{1}{2}}\right).$$

Thus $|\psi_{u,v}(s_N) - q(u, v)| \leqslant \frac{\kappa_p}{N}$, which ends the proof. ∎

We can now state the main result of this section.

Theorem 2.7. There exists a β_0 such that if $\beta \leqslant \beta_0$, then

$$U_N \equiv \sum_{u, v \leqslant d} \nu((R^{1,2}(u, v) - q(u, v))^2) \leqslant \frac{\kappa_{d,p}}{N},$$

$$V_N \equiv \sum_{u, v \leqslant d} \nu((R(u, v) - s(u, v))^2) \leqslant \frac{\kappa_{d,p}}{N}.$$

Proof. This proof is borrowed from ref. 12, Theorem 2.11.1, and is included here for sake of completeness: using symmetry among sites, we have $U_N = \nu(f)$, with

$$f = \sum_{u, v \leqslant d} (\varepsilon^1(u)\, \varepsilon^2(v))(R^{1,2}(u, v) - q(u, v)).$$

Moreover

$$f^2 \leqslant d^2 \sum_{u, v \leqslant d} (R^{1,2}(u, v) - q(u, v))^2,$$

and setting

$$R_-^{1,2}(u, v) = \frac{1}{N} \sum_{i \leqslant N-1} \sigma_i^1(u)\, \sigma_i^2(v),$$

it is easily checked that

$$|R^{1,2}(u, v) - R_-^{1,2}(u, v)| \leqslant \frac{d}{N}.$$

Hence, appealing to Lemma 2.6, we get

$$|\nu_0(f)| \leqslant \frac{\kappa_{d,p}}{N}.$$

On the other hand, a direct applicaction of Proposition 2.3 yields $\nu_t(f) \leqslant \kappa_{d,p} \nu(f)$ for all positive functions $f\colon (S^N)^n \to \mathbb{R}$. Combining this result with Proposition 2.5 and the fact that $\nu(f) \leqslant \nu_0(f) + \sup_{t \in [0,1]} |\nu_t'(f)|$, we get

$$U_N = \nu(f) \leqslant \kappa_{d,p} \left(\frac{1}{N} + \beta^2 U_N^{1/2} (U_N^{1/2} + V_N^{1/2}) \right).$$

The same kind of arguments lead to the inequality

$$V_N \leqslant \kappa_{d,p} \left(\frac{1}{N} + \beta^2 V_N^{1/2} (U_N^{1/2} + V_N^{1/2}) \right),$$

from which our claim follows. ∎

Remark 2.8. In all the previous computations, we did not stress the dependence of the quantities we manipulated upon the reference measure μ. However, many of our estimates just relied on the fact that the support of μ is the bounded set S, and that μ is a measure of unit mass. Thus, our estimates will be uniform in μ, which means in particular that the constants $\kappa_{d,p}$ in Theorem 2.7, and the estimate (14) on the contraction property of the function Φ defining Q and S, are independent of μ.

3. LARGE DEVIATIONS PRINCIPLE

In this Section, we will prove, for a quenched disorder g, an almost sure large deviations principle for the overlap $R^{1,2}$ of two independent configurations under the Gibbs measure G_N of the one dimensional p-spins system introduced at Section 1. This large deviations result will be a consequence of the application of the Gärtner–Ellis Theorem, and thus, the natural quantity under consideration, trivially defined for all $\lambda \in \mathbb{R}$, will be

$$\Lambda_N(\lambda) = \log(\langle \exp(\lambda R^{1,2}) \rangle_*), \tag{18}$$

where the notation $\langle \cdot \rangle_*$ has been introduced at (1). Our first task will be to describe the asymptotic behavior of Λ_N as a function of λ.

Proposition 3.1. Let Λ_N be the function defined by (18). Then, almost surely, we have, for all $\lambda \in \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \Lambda_N(N\lambda) = \Lambda(\lambda),$$

where $\Lambda(\lambda)$ can be expressed as $\Lambda(\lambda) = \int_0^\lambda s_\eta(1,2)\, d\eta$, and s_η is the solution to (11), where the average $\mathbf{Av}$ is based on a measure μ_η on $S = \{\varepsilon \in \mathbb{R}^2; [\varepsilon(1)]^2 + [\varepsilon(2)]^2 = 2\}$ given by

$$\mu_\eta(\varepsilon(1), \varepsilon(2)) = \frac{\sum_{j,k\in\{-1,+1\}} \exp(\eta jk)\, \delta_{(j,k)}(\varepsilon(1), \varepsilon(2))}{4\cosh(4\eta)}. \tag{19}$$

Proof. This proof will be divided into several steps

Step 1. For $N \geqslant 1$, and $\lambda \in \mathbb{R}$, set $\varphi_N(\lambda) = \mathbf{E}[N^{-1}\Lambda_N(N\lambda)]$, that is

$$\varphi_N(\lambda) = \frac{1}{N}\mathbf{E}\left[\log\left(\left\langle \exp\left(\lambda \sum_{i\leqslant N} \sigma_i^1 \sigma_i^2\right)\right\rangle_*\right)\right].$$

Then it is easily seen that φ_N is a differentiable function, and that

$$\varphi_N'(\lambda) = \mathbf{E}\left[\frac{D_\lambda}{N_\lambda}\right],$$

with

$$D_\lambda = \frac{1}{N}\left\langle \sum_{i \leqslant N} \sigma_i^1 \sigma_i^2 \exp\left(\lambda \sum_{i \leqslant N} \sigma_i^1 \sigma_i^2\right)\right\rangle_*$$

$$N_\lambda = \left\langle \exp\left(\lambda \sum_{i \leqslant N} \sigma_i^1 \sigma_i^2\right)\right\rangle_*.$$

Note then that in the last expression, we have

$$D_\lambda = \frac{1}{N} \sum_{\sigma^1, \sigma^2 \in \Sigma_N} \sum_{i \leqslant N} \sigma_i^1 \sigma_i^2 \exp\left(\lambda \sum_{i \leqslant N} \sigma_i^1 \sigma_i^2\right) \exp(-\beta H_N(\sigma^1) - \beta H_N(\sigma^2)).$$

To make the link with the notations of Section 2, set $\sigma_i = (\sigma_i^1, \sigma_i^2) = (\sigma_i(1), \sigma_i(2))$ for $i \leqslant N$. Then D_λ can be written as

$$D_\lambda = \frac{(4\cosh(\lambda))^N}{N} \sum_{i \leqslant N} \int_{S^N} \sigma_i(1)\, \sigma_i(2) \exp(-\beta H_N(\sigma))\, d\mu_\lambda^{\otimes N}(\sigma),$$

where H_N is defined by (2) for S taken as the ball of radius $2^{1/2}$ in $\mathbb{R}^2$, and μ_λ as in (19). Thus, taking up the notations of Section 2, we have

$$\varphi_N'(\lambda) = \mathbf{E}[\langle R(1,2)\rangle].$$

In order to stress the dependence of this quantity upon λ, we will write

$$\varphi_N'(\lambda) = \mathbf{E}[\langle R(1,2)\rangle_\lambda].$$

Step 2. We have shown in Theorem 2.7 that for all $\lambda \in \mathbb{R}$,

$$\lim_{N\to\infty} \varphi_N'(\lambda) = \lim_{N\to\infty} \mathbf{E}[\langle R(1,2)\rangle_\lambda] = s_\lambda(1,2).$$

Moreover, since $R(1,2)$ is a quantity bounded by 2 almost surely, the dominated convergence theorem can be applied, and noting that $\varphi_N(0) = 0$, we get, for all $\lambda \in \mathbb{R}$,

$$\lim_{N\to\infty} \varphi_N(\lambda) = \int_0^\lambda s_\eta(1,2)\, d\eta = \Lambda(\lambda).$$

Furthermore, since all the estimates of Section 2 are uniform with respect to the reference measure μ (see Remark 2.8), we have

$$|\varphi_N(\lambda) - \Lambda(\lambda)| \leqslant \frac{\kappa_{d,p}\,|\lambda|}{N}. \tag{20}$$

Step 3. For a fixed $\lambda \in \mathbb{R}$, let us study now the almost sure behavior of the random variable $X_N(\lambda) = \frac{1}{N}\Lambda_N(N\lambda)$. Setting again $\sigma = (\sigma^1, \sigma^2) \in \Sigma_N^2$, we have

$$[X_N(\lambda)](g) = \frac{1}{N}\left\{\log\left(\sum_{\sigma\in\Sigma_N^2} \exp(a(\sigma)\cdot g + b_\lambda(\sigma))\right) - \log\left(\sum_{\sigma\in\Sigma_N^2} \exp(a(\sigma)\cdot g)\right)\right\},$$

where $g = \{g_I;\, I \in A_N\}$ and

$$a(\sigma) = \{\beta u_N(\eta_I^1 + \eta_I^2);\, I \in A_N\}, \qquad b_\lambda(\sigma) = \lambda \sum_{i \leqslant N} \sigma_i^1 \sigma_i^2,$$

with $\eta_I^l = \sigma_{i_1}^l \cdots \sigma_{i_p}^l$ for all $I = (i_1,..., i_p \in A_N)$. Now, for the Euclidian norm $\|\cdot\|$ taken in $\mathbb{R}^{|A_N|}$, we have

$$\|a(\sigma)\| \leqslant \beta u_N(2\,|A_N|)^{1/2} \leqslant \beta N^{1/2},$$

which immediately gives, for two vectors $g, \hat{g} \in \mathbb{R}^{|A_N|}$,

$$|[X_N(\lambda)](\hat{g}) - [X_N(\lambda)](g)| \leqslant \frac{\beta}{N^{1/2}}\,\|\hat{g} - g\|.$$

Thus, by the classical concentration inequality for Gaussian vectors, we have, for all $u > 0$,

$$P(|X_N(\lambda) - \varphi_N(\lambda)| \geqslant u) \leqslant 2\exp\left(-\frac{Nu^2}{4\beta^2}\right).$$

Hence, invoking (20), and taking $u = N^{-1/4}$, we get

$$P(|X_N(\lambda) - \Lambda(\lambda)| \geqslant N^{-1/4} + \kappa_{d,p}\,|\lambda|\,N^{-1}) \leqslant 2\exp\left(-\frac{N^{1/2}}{4\beta^2}\right),$$

which gives

$$\text{a.s.} \quad -\lim_{N\to\infty} X_N(\lambda) = \Lambda(\lambda),$$

by a simple application of the Borel–Cantelli Lemma.

Step 4. We should now check that almost surely, $X_N(\lambda)$ tends to $\Lambda(\lambda)$ for all $\lambda \in \mathbb{R}$. This is certainly true for all $\lambda \in \mathbb{Q}$. Now, it is easily checked that $\lambda \mapsto X_N(\lambda)$ is a differentiable function on $\mathbb{R}$, and that

$$X'_N(\lambda) = \langle R(1,2)\rangle.$$

Hence, $X'_N(\lambda)$ is trivially bounded by 2, and since Λ is also a continuous function, we get the desired convergence. ∎

For $x \in \mathbb{R}$, set now

$$\Lambda^*(x) = \sup\{\lambda x - \Lambda(\lambda);\ \lambda \in \mathbb{R}\}. \tag{21}$$

Turning to the main point of this paper, we get the following result:

Theorem 3.2. There exists a $\beta_0 > 0$ such that, for all $\beta \leqslant \beta_0$, for almost every realization of the disorder g, we have

(i) For any closed set $F \subset \mathbb{R}$,

$$\limsup_{N\to\infty} \frac{1}{N} \log(\langle \mathbf{1}_F(R^{1,2})\rangle_*) \leqslant -\inf\{\Lambda^*(x);\ x \in F\}.$$

(ii) For any open set $G \subset \mathbb{R}$,

$$\liminf_{N\to\infty} \frac{1}{N} \log(\langle \mathbf{1}_G(R^{1,2})\rangle_*) \geqslant -\inf\{\Lambda^*(x);\ x \in G\}.$$

Proof. Note that $D_\Lambda \equiv \{\lambda;\ \Lambda(\lambda) < \infty\} = \mathbb{R}$. The bounds for F and G given by the Gärtner–Ellis Theorem (we refer to ref. 13 for an account on this Theorem as well as for all further notations and concepts on the large deviations principle) are

$$\limsup_{N\to\infty} \frac{1}{N} \log(\langle \mathbf{1}_F(R^{1,2})\rangle_*) \leqslant -\inf\{\Lambda^*(x);\ x \in F\}$$

$$\liminf_{N\to\infty} \frac{1}{N} \log(\langle \mathbf{1}_G(R^{1,2})\rangle_*) \geqslant -\inf\{\Lambda^*(x);\ x \in G \cap \mathscr{F}\},$$

where $\mathscr{F}$ is the set of exposing points of Λ. We will show now that $(-1; 1) \subset \mathscr{F}$. From ref. 13, Lemma 2.3.9, if there exists a given $\lambda \in \mathbb{R}$ such that $x = \Lambda'(\lambda)$, then $x \in \mathscr{F}$. Let us show then that this happens for all $x \in (-1; 1)$.

We have seen that Λ is differentiable on $\mathbb{R}$, and that $\Lambda'(\lambda) = s_\lambda(1, 2)$. But the equation (11) defining (Q, S) involves a function Φ (see definition (13)) which is differentiable with respect to Q, S and λ (the first two claims have been shown in the proof of Proposition 2.4 and the third one follows by the same kind of arguments), and contracting with respect to Q and S for small enough β, uniformly in $\lambda \in \mathbb{R}$ (see Remark 2.8). This yields easily the continuity of Λ' on $\mathbb{R}$.

Let us investigate now the asymptotic behavior of Λ': going back to Eqs. (10)–(12), we can write

$$s_\lambda(1, 2) = \mathbf{E}\left[\frac{D_\lambda}{N_\lambda}\right],$$

where, setting $\ell(w, z) = [s_\lambda(w, z)]^{p-1} - [q_\lambda(w, z)]^{p-1}$ and $\alpha = (p/2)^{1/2}\,\beta$, D_λ and N_λ are given by

$$\begin{aligned} D_\lambda = & \sum_{\varepsilon(1),\, \varepsilon(2) \in \{-1; 1\}} \exp(\lambda\varepsilon(1)\,\varepsilon(2))\,\varepsilon(1)\,\varepsilon(2) \\ & \times \exp\left(\alpha \sum_{w \leqslant 2} Y(w)\,\varepsilon(w) + \frac{\alpha^2}{2} \sum_{w, z \leqslant 2} \varepsilon(w)\,\varepsilon(z)\,\ell(w, z)\right) = A_\lambda - B_\lambda, \end{aligned}$$

with

$$A_\lambda = 2\exp\left(\frac{\alpha^2}{2}(\ell(1, 1) + 2\ell(1, 2) + \ell(2, 2)) + \lambda\right)\cosh(\alpha(Y(1) + Y(2)))$$

$$B_\lambda = 2\exp\left(\frac{\alpha^2}{2}(\ell(1, 1) - 2\ell(1, 2) + \ell(2, 2)) - \lambda\right)\cosh(\alpha(Y(1) - Y(2))),$$

and

$$N_\lambda = A_\lambda + B_\lambda.$$

Then, it is easily checked that

$$\text{a.s.} \quad -\lim_{\lambda \to \infty} \frac{D_\lambda}{N_\lambda} = 1, \qquad \text{a.s.} \quad -\lim_{\lambda \to -\infty} \frac{D_\lambda}{N_\lambda} = -1.$$

By dominated convergence, we hence get that

$$\lim_{\lambda \to \infty} \Lambda'(\lambda) = 1, \qquad \lim_{\lambda \to -\infty} \Lambda'(\lambda) = -1.$$

The continuity of Λ' yields now $(-1; 1) \subset \mathscr{F}$. On the other hand, one can also easily verify that

$$\lambda \mapsto \frac{A_\lambda - B_\lambda}{A_\lambda + B_\lambda} = \frac{1 - \frac{A_\lambda}{B_\lambda}}{1 + \frac{A_\lambda}{B_\lambda}}$$

is a strictly increasing function. Hence $|\Lambda'(\lambda)| < 1$ for all $\lambda \in \mathbb{R}$. This means, by definition (21) of Λ^*, that $\Lambda^*(x) = +\infty$ whenever $|x| \geqslant 1$. Hence,

$$\inf\{\Lambda^*(x); x \in G \cap \mathscr{F}\} \leqslant \inf\{\Lambda^*(x); x \in G \cap (-1; 1)\} = \inf\{\Lambda^*(x); x \in G\},$$

which ends the proof. ∎

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